## II. Relating to the Definition of a Function $E$.

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A function $E$ is defined by the following three statements:

1. $E_{[r]}^{(l,n,x)} = 1 + \dfrac{x}{l} + \dfrac{x^2}{l(l+n)} + \cdots + \dfrac{x^r}{l(l+n)\cdots(l+\overline{r-1}n)}$.

2. $E^{[r]}_{(l,n,x)} = -\dfrac{l-n}{x} - \dfrac{(l-n)(l-2n)}{x^2} - \cdots$
$$- \frac{(l-n)(l-2n)\cdots(l-\overline{r-1}n)}{x^{r-1}}.$$

3. $E^{(x^n)} = 1 + \dfrac{nx^n}{1} + \dfrac{(nx^n)^2}{1+n} + \dfrac{(nx^n)^3}{(1+n)(1+2n)} + \cdots$.

The second statement is a consequence from the first when $r$ is negative; the third a special definition for the purpose of simplification in the present remarks. The index $r$ in the first and second may be omitted when for particular values of $l$ and $n$ the series either terminates or becomes infinite; the index $l$ may also be omitted when $l = 1$; thus, $E^{(1,x)} = e^x$.

It will now appear by reference to the September, 1917, number of the Monthly, pp. 342–343, that[1] $\int \sin\theta^2 d\theta$ and $\int \cos\theta^2 d\theta$ may be written in the following concise forms:

$$\begin{aligned}
\int \sin\theta^2 d\theta &= -\theta \sum_{t=0}^{\infty} \prod_{s=1}^{s=t} \frac{2\theta^2}{1+2s} \cdot \sin\left(t\frac{\pi}{2} - \theta^2\right) \\
&= \theta \sin\theta^2 \left(1 - \frac{(2\theta^2)^2}{3\cdot 5} + \frac{(2\theta^2)^4}{3\cdot 5\cdot 7\cdot 9} - + \cdots\right) \\
&\qquad - \theta\cos\theta^2\left(\frac{2\theta^2}{3} - \frac{(2\theta^2)^3}{3\cdot 5\cdot 7} + \frac{(2\theta^2)^5}{3\cdot 5\cdots 11} - + \cdots\right) \\
&= \tfrac{1}{4}i\theta^{-1}\sin\theta^2(E^{(-i\theta^2)} - E^{(i\theta^2)}) + \tfrac{1}{4}\theta^{-1}\cos\theta^2(E^{(-i\theta^2)} + E^{(i\theta^2)}) \\
&\qquad - \tfrac{1}{2}\theta^{-1}\cos\theta^2 \\
&= \tfrac{1}{4}\theta^{-1}(\cos\theta^2 + i\sin\theta^2)E^{(-i\theta^2)} + \tfrac{1}{4}\theta^{-1}(\cos\theta^2 - i\sin\theta^2)E^{(i\theta^2)} \\
&\qquad - \tfrac{1}{2}\theta^{-1}\cos\theta^2 \\
&= \tfrac{1}{4}\theta^{-1}(e^{i\theta^2}E^{(-i\theta^2)} + e^{-i\theta^2}E^{(i\theta^2)}) - \tfrac{1}{2}\theta^{-1}\cos\theta^2; \\
\int \cos\theta^2 d\theta &= \tfrac{1}{4}i\theta^{-1}(e^{i\theta^2}E^{(-i\theta^2)} - e^{-i\theta^2}E^{(i\theta^2)}) + \tfrac{1}{2}\theta^{-1}\sin\theta^2.
\end{aligned}$$

If in a similar manner the integral of $e^{\theta^2}d\theta$ is sought by means of the reduction

[1] We write $\sin\theta^2$ for $\sin(\theta^2)$, etc.

formula

$$\int \theta^m e^{\theta^n} d\theta = \frac{1}{n}\,\theta^{m-n+1}e^{\theta^n} - \frac{m-n+1}{n}\int \theta^{m-n}e^{\theta^n}d\theta,$$

the following two forms are obtained:

$$\int \theta^m e^{\theta^n} d\theta = \frac{1}{n}\,\theta^{m-n+1}e^{\theta^n}\sum_{t=0}^{t=r}\prod_{s=1}^{s=t}\frac{sn-m-1}{n\theta^n} + \prod_{s=1}^{s=r+1}\frac{sn-m-1}{n}\cdot\int \theta^{m-(r+1)n}e^{\theta^n}d\theta,$$

$$\int \theta^m e^{\theta^n} d\theta = -\frac{1}{n}\,\theta^{m-n+1}e^{\theta^n}\sum_{t=1-r}^{t=-1}\prod_{s=t+1}^{s=0}\frac{n\theta^n}{sn-m-1}$$

$$+\prod_{s=2-r}^{s=0}\frac{n}{sn-m-1}\cdot\int \theta^{m+(r-1)n}e^{\theta^n}d\theta,$$

where $r$ is a positive integer.

The second of these gives, for $m = 0$, $n = 2$, the desired development as a convergent series, which by 3. takes the form

$$\int e^{\theta^2}d\theta = -\tfrac{1}{2}\theta^{-1}e^{\theta^2}\left(-\frac{2\theta^2}{1}+\frac{(2\theta^2)^2}{3}-\frac{(2\theta^2)^3}{3\cdot 5}+-\cdots\right)$$

$$= -\tfrac{1}{2}\theta^{-1}e^{\theta^2}E^{(-\theta^2)} + \tfrac{1}{2}\theta^{-1}e^{\theta^2}.$$

Comparison of these three results shows the following relation to hold:

$$\int \cos\theta^2 d\theta + i\int \sin\theta^2 d\theta = \int e^{i\theta^2}d\theta,$$

as was to be expected.

When $m+1$ is a multiple of $n$, the development of $\int \theta^m e^{\theta^n}d\theta$ is a finite series. Thus, let $m = 7, 0$; $n = 2, \frac{1}{4}$, respectively, and $r = 3$, and the first formula gives:

$$\int \theta^7 e^{\theta^2}d\theta = \tfrac{1}{2}\theta^6 e^{\theta^2}\sum_{t=0}^{t=3}\prod_{s=1}^{s=t}(s-4)\theta^{-2}$$

$$= -3e^{\theta^2}\left(1-\frac{\theta^2}{1!}+\frac{\theta^4}{2!}-\frac{\theta^6}{3!}\right)$$

$$= \tfrac{1}{8}\theta^8 e^{\theta^2}E_{(1/5,\,-1/5\theta^2)} \qquad \text{[by 2];}$$

$$\int e^{\theta^{1/4}}d\theta = 4\theta^{3/4}e^{\theta^{1/4}}\sum_{t=0}^{t=3}\prod_{s=1}^{s=t}(s-4)\theta^{-1/4}$$

$$= -4!\,e^{\theta^{1/4}}\left(1-\frac{\theta^{1/4}}{1!}+\frac{\theta^{2/4}}{2!}-\frac{\theta^{3/4}}{3!}\right)$$

$$= \theta e^{\theta^{1/4}}E_{(1/5,\,-1/5\theta^{1/4})} \qquad \text{[by 2].}$$

It is interesting to observe that these values of $m$ and $n$ substituted in the second of the formulae above will give, after a little transformation, the same finite

forms; thus, in the latter case:

$$\int e^{\theta^{1/4}} d\theta = -4!\, e^{\theta^{1/4}}\left(1 - \frac{\theta^{1/4}}{1} + \frac{\theta^{2/4}}{2!} - \frac{\theta^{3/4}}{3!}\right) + 4!,$$

differing from the previous result only by a constant term.

This result is general, and gives, when the exponent of the argument is a unit fraction:

$$\int e^{\theta^{1/n}} d\theta = (-1)^{n-1} n!\, e^{\theta^{1/n}} \sum_{t=0}^{t=n-1} (-1)^t \frac{\theta^{t/n}}{t!},$$

or, the same expression plus the term $(-1)^n n!$.

## III. Relating to Magic Squares for the New Year, 1918.

By S. A. Corey, Albia, Iowa.

The two following $4 \times 4$ magic squares are taken indiscriminately one from each of two large families of $4 \times 4$ magic squares with sum 1918 which may be formed in like manner from the two sets of 15 square numbers here employed. It is improbable that other families of $4 \times 4$ magic squares with the sum 1918 can be found with similar characteristics.

| | | | |
|---|---|---|---|
| 464 | 383 | 513 | 558 |
| 591 | 480 | 482 | 365 |
| 414 | 429 | 527 | 548 |
| 449 | 626 | 396 | 447 |

(*A*)

| | | | |
|---|---|---|---|
| 461 | 409 | 516 | 532 |
| 565 | 483 | 508 | 362 |
| 440 | 426 | 501 | 551 |
| 452 | 600 | 393 | 473 |

(*B*)

Some of the peculiarities of the *families* of magic squares from which (*A*) and (*B*) are taken are as follows:

Each number employed may be decomposed into the sum of the squares of *four* separate numbers taken from the series of natural numbers, 1 to 18 inclusive (18 being the last two figures of 19*18*).

In forming such sums *all* the square numbers used are used just *four* times with *one* exception in each case as follows:

In (*A*) 10 is used *eight* times, and
In (*B*) 6 is used *eight* times.

In forming the numbers employed in (*A*) all the natural numbers, 1 to 18 inclusive, are used, except 1, 11 and 13, the sum of which is 25.

In forming the numbers employed in (*B*) all the natural numbers, 1 to 18 inclusive, are used, except 5, 9 and 11, the sum of which is likewise 25. To illustrate:

$2^2 + 6^2 + 10^2 + 18^2 = 464$, $3^2 + 7^2 + 10^2 + 15^2 = 383$, $4^2 + 8^2 + 12^2 + 17^2 = 513$, $5^2 + 9^2 + 14^2 + 16^2 = 558$ [464, 383, 513 and 558 being the numbers in